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COMPARISON THEOREMS FOR HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS OF GENERAL ORDER.*

BY R. D. CARMICHAEL.

Introduction.

The fundamental notion of adjoint systems of boundary conditions associated with adjoint linear differential equations of order n, first defined in a general manner by Birkhoff,† is fruitful of interesting consequences concerning the relation between solutions of two linear differential equations. In particular it leads to certain extensions of one of Sturm's classical theorems of comparison‡ to equations of order higher than the second. The purpose of this paper is to set forth some of the results which thus emerge. The theorems obtained may be compared with certain other related results which are likewise of the nature of Sturmian theorems.§

After deriving preliminary formulæ concerning Lagrange adjoint differential equations in § 1 and concerning Birkhoff adjoint linear forms in § 2 (for which no essential novelty is claimed), I give in § 3 the two general theorems of comparison of the paper. These are of considerable generality and may be employed in an easy way to derive numerous more special theorems of comparison. In particular, it is pointed out that the first of these general theorems contains as a special case Sturm's classical theorem of comparison for linear homogeneous binomial equations of the second order. In § 4 I give four of the more elegant specific consequences of the general theorems of comparison. Finally, in § 5 I consider briefly a case in which one of the two equations involved has special form and show by an example how the theorem obtained may be employed to yield information concerning the zeros of solutions of certain types of equations. The general treatment of the matter raised by this example is reserved for later consideration.

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* Presented to the American Mathematical Society (Chicago), April, 1917.

† G. D. Birkhoff, Transactions of the American Mathematical Society, vol. 9 (1908), pp. 373–395. See also, in the same journal, a further treatment of adjoint systems by M. Bôcher, vol. 14 (1913), pp. 403–420, and by D. Jackson, vol. 17 (1916), pp. 418–424. As a matter of fact it is possible to state our principal results without reference to Birkhoff's notion of adjoint conditions; but that notion was the guide in the investigation and illuminates the results obtained.

‡ C. Sturm, Journal de mathématiques, vol. 1 (1836), pp. 106–186. See also M. Bôcher, Bulletin of the American Mathematical Society, ser. 2, vol. 4 (1898), pp. 295–313, 365–376.

§ See G. D. Birkhoff, *Annals of Mathematics*, vol. 12 (1911), pp. 103–127; A. Davidoglou, Annales de l'École Normale Supérieure, ser. 3, vol. 22 (1905), pp. 539–565; O. Haupt, Dissertation, Würzburg, 1911.

§ 1. Preliminary formulæ.

In connection with the linear differential expression

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u \tag{1}$$

let us consider also the Lagrange adjoint expression

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n} (l_n v) + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}} (l_{n-1} v) + \cdots + \frac{d}{dx} (l_1 v) + l_0 v.$$

Arranging the latter according to the derivatives of v we write it in the form

$$M(v) \equiv m_n \frac{d^n v}{dx^n} + m_{n-1} \frac{d^{n-1} v}{dx^{n-1}} + \cdots + m_1 \frac{dv}{dx} + m_0 v, \tag{2}$$

where

$$\begin{aligned}
(-1)^n m_n &= l_n, \\
(-1)^{n-1} m_{n-1} &= l_{n-1} - n l_n', \\
(-1)^{n-2} m_{n-2} &= l_{n-2} - (n-1) l_{n-1}' + \frac{(n-1)n}{2!} l_n'', \\
(-1)^{n-3} m_{n-3} &= l_{n-3} - (n-2) l_{n-2}' \\
&\qquad + \frac{(n-2)(n-1)}{2!} l_{n-1}'' - \frac{(n-2)(n-1)n}{3!} l_n''', \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .
\end{aligned} \tag{3}$$

Here we assume that the functions* $l_n, l_{n-1}, \cdots, l_1, l_0$ are continuous in the (finite) interval defined by the inequalities $a \leqq x \leqq b$ and that l_n is of one sign, say positive, in (ab). Moreover, we suppose that the derivatives of l_i, $i = 1, 2, \cdots, n$, exist up to order i inclusive† and that these are continuous in (ab).

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* All functions employed in the paper are assumed to be real-valued single-valued and continuous functions of the real variable x.

† As Bôcher has pointed out (l. c., p. 415), the demand for the existence and the continuity of the derivatives of the l's may be omitted in case one considers a system of n forms of the first order in place of the single form $L(u)$ of the nth order. (Compare also Bôcher, l. c., p. 404, first footnote.)

If we write

$$\begin{aligned} R(u,\,v) \equiv\ & [l_nv]\frac{d^{n-1}u}{dx^{n-1}} + \left[l_{n-1}v - \frac{d}{dx}(l_nv) \right]\frac{d^{n-2}u}{dx^{n-2}} \\ & + \left[l_{n-2}v - \frac{d}{dx}(l_{n-1}v) + \frac{d^2}{dx^2}(l_nv) \right]\frac{d^{n-3}u}{dx^{n-3}} + \cdots \\ & + \left[l_1v - \frac{d}{dx}(l_2v) + \frac{d^2}{dx^2}(l_3v) - \cdots \right. \\ & \left. + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(l_nv) \right]u, \end{aligned} \tag{4}$$

then it is well known* and easily verified that

$$vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u,\,v)\}. \tag{5}$$

We now introduce also the forms

$$P(y) \equiv p_n\frac{d^ny}{dx^n} + p_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_1\frac{dy}{dx} + p_0y \tag{6}$$

and

$$D(y) \equiv P(y) - M(y). \tag{7}$$

Here we assume that the functions p_n, p_{n-1}/p_n, p_{n-2}/p_n, $\cdots$, p_0/p_n of x are continuous in (ab).

We shall uniformly employ the notation u_k, v_k, y_k, for various subscripts k, to denote particular solutions of the differential equations

$$L(u) = 0, \qquad M(v) = 0, \qquad P(y) = 0, \tag{8}$$

respectively. From the relations $L(u_s) = 0$ and $P(y_t) = 0$ we have $y_tL(u_s) - u_sP(y_t) = 0$; whence, in view of (5) and (7), it follows readily that

$$\frac{d}{dx}\{R(u_s,\,y_t)\} - D(y_t)u_s = 0. \tag{9}$$

Thence, if α and β are in (ab), we have

$$R(u_s,\,y_t)\,\Big|_{x=\alpha}^{x=\beta} = \int_\alpha^\beta D(y_t)u_sdx. \tag{10}$$

§ 2. **Adjoint linear forms associated with adjoint linear differential equations.**

Let us denote the kth derivative with respect to x of any function g by $g^{(k)}$ with the understanding that $g^{(0)} \equiv g$. Let α and β be any two points in (ab) and take $\alpha < \beta$. By $W_1, W_2, \cdots, W_{2n}$ we denote any $2n$ linearly independent homogeneous forms

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* See, for instance, Forsyth's Theory of Differential Equations, vol. IV, p. 252.

$$W_i \equiv W_i(u) = \sum_{j=1}^{n} c_{ij} u^{(j-1)}(\alpha) + \sum_{j=n+1}^{2n} c_{ij} u^{(j-n-1)}(\beta), \quad i = 1, 2, \cdots, 2n, \tag{11}$$

in the $2n$ arguments $u(\alpha), u'(\alpha), \cdots, u^{(n-1)}(\alpha), u(\beta), u'(\beta), \cdots, u^{(n-1)}(\beta)$. We denote by $\bar{D}$ the determinant of these forms:

$$\bar{D} = |c_{ij}|.$$

With the linear forms (11) Birkhoff (l. c., p. 375) has associated a second set of linear homogeneous forms $V_1, V_2, \cdots, V_{2n}$ in the same $2n$ arguments which we may call the *set adjoint to* $W_1, W_2, \cdots, W_{2n}$ *with respect to the differential expression* $L(u)$ *and the points* α *and* β. They are defined by the relation

$$\sum_{i=1}^{2n} W_i(u) V_i(y) \equiv R(u, y) \Big|_{x=\alpha}^{x=\beta} \tag{12}$$

which is to subsist as an identity in u and y and their derivatives at the points α and β.

It is clear that the second member of (12) is a bilinear form in the two sets of variables $u(\alpha), u'(\alpha), \cdots, u^{(n-1)}(\alpha), u(\beta), u'(\beta), \cdots, u^{(n-1)}(\beta)$ and $y(\alpha), y'(\alpha), \cdots, y^{(n-1)}(\alpha), y(\beta), y'(\beta), \cdots, y^{(n-1)}(\beta)$. In view of (4) we may write

$$R(u, y) \Big|_{x=\alpha}^{x=\beta} = \sum_{i,j=1}^{n} r_{ij} u^{(i-1)}(\alpha) y^{(j-1)}(\alpha) + \sum_{i,j=n+1}^{2n} r_{ij} u^{(i-n-1)}(\beta) y^{(j-n-1)}(\beta), \tag{13}$$

where the coefficients r_{ij} are independent of u and y but in the first summation are functions of α and in the second are functions of β. The quantities r_{ij}, $i, j = 1, 2, \cdots, 2n$, not entering into (13), we take to be zero. Then we see that $r_{ij} = 0$ when either subscript is greater than n and the other is not greater than n. Also, from (4) it follows that $r_{ij} = 0$ if $i \leqq n$, $j \leqq n$ and $i + j > n + 1$; and again that $r_{ij} = 0$ if $i > n$, $j > n$ and $i + j - 2n > n + 1$. By R we denote the determinant of the bilinear form (13); we have

$$R = |r_{ij}| = \{l_n(\alpha) l_n(\beta)\}^n, \tag{14}$$

a value which is different from zero since l_n does not vanish in $(\alpha\beta)$.

That the forms V are indeed uniquely determined by (12) follows from the fact that $\bar{D} \neq 0$. We denote them by

$$V_i(y) = \sum_{j=1}^{n} \gamma_{ij} y^{(j-1)}(\alpha) + \sum_{j=n+1}^{2n} \gamma_{ij} y^{(j-n-1)}(\beta), \qquad i = 1, 2, \cdots, 2n, \tag{15}$$

and their determinant by Δ:

$$\Delta = |\gamma_{ij}|.$$

It is easy to see that we have

$$\sum_{i=1}^{2n} c_{ij}\gamma_{is} = r_{js}, \qquad j, s = 1, 2, \cdots, 2n. \tag{16}$$

Thence it follows that $\bar{D}\Delta = R$. Hence $\Delta \neq 0$ and the forms V are therefore linearly independent. Clearly, then, the forms $W_1, \cdots, W_{2n}$ are adjoint to the forms $V_1, \cdots, V_{2n}$, so that the relation between the W's and the V's is reciprocal. Moreover, the coefficients γ_{is} may be determined from (16) in terms of the coefficients c_{ij} and *vice versa.*

In accordance with Birkhoff's definition (l. c., p. 375), as extended by Bôcher (l. c., p. 418), we shall say that the two sets of boundary conditions

$$W_\sigma(u) = 0, \qquad V_\rho(y) = 0, \tag{17}$$

where σ runs over m numbers of the set $1, 2, \cdots, 2n$ and ρ runs over the remaining $2n - m$ numbers of this set, are *adjoint with respect to the differential expression $L(u)$ and the points α and β.* The most important case, namely that in which $m = n$, is the one of Birkhoff's original definition.

In the case when $m = n$, if an interchange of y and u in (17) merely interchanges the two sets of conditions* we say that the boundary conditions $W = 0$ in (17) are *self-adjoint.* If an interchange of y and u and of α and β in (17) interchanges the two sets of conditions* we say that the boundary conditions $W = 0$ in (17) are *skew-self-adjoint.*†

It is usually desirable to choose the notation so that σ in (17) runs over the first m or the last m integers of the set $1, 2, \cdots, 2n$ and ρ over the remaining $2n - m$ of these integers. If we assume for the moment that σ runs over the set $1, 2, \cdots, m$ we shall see from (16) that the second set of conditions in (17) depends only on the conditions $W_1 = 0, \cdots, W_m = 0$ and not at all on the remaining W's, in the sense that a variation in the choice of $W_{m+1}, \cdots, W_{2n}$ merely replaces the conditions $V_{m+1} = 0, \cdots, V_{2n} = 0$ by a set of equivalent conditions. Moreover, if the conditions $W_1 = 0, \cdots, W_m = 0$ are replaced by any set of equivalent conditions the resulting set of conditions $V_{m+1} = 0, \cdots, V_{2n} = 0$ are again equivalent to the old. For, in view of the conditions $W_1 = 0, \cdots, W_m = 0$ the suggested changes are all allowed for by replacing c_{ij} by c_{ij}', where

$$c_{ij}' = \sum_{k=1}^{m} d_{ik}c_{kj}, \qquad i = 1, 2, \cdots, m;$$

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* It is not necessary that the individual equations shall be preserved; it is sufficient if the conditions are replaced by equivalent conditions.

† I believe that the notion of skew-self-adjoint conditions has not been employed before.

$$c_{ij}' = \sum_{k=m+1}^{2n} d_{ik} c_{kj}, \qquad i = m+1, \cdots, 2n.$$

In view of (16) we see that this is equivalent to replacing γ_{is} by γ_{is}' where

$$\gamma_{is}' = \sum_{k=1}^{m} d_{ki}\gamma_{ks}, \qquad i = 1, 2, \cdots, m;$$

$$\gamma_{is}' = \sum_{k=m+1}^{2n} d_{ki}\gamma_{ks}, \qquad i = m+1, \cdots, 2n.$$

We now define a particular pair of adjoint forms special cases of which we shall employ later: Let σ and τ be two integers (the same or different) of the set $0, 1, 2, \cdots, n-1$. Then for the W's we take the following forms:

$$\left.\begin{array}{l} W_i(u) = u^{(i-1)}(\alpha) - k_i u^{(\sigma)}(\alpha), \qquad 1 \leqq i \leqq n, \qquad i \neq \sigma + 1; \\ \qquad\qquad\qquad\qquad W_{\sigma+1}(u) = u^{\sigma}(\alpha) - k u^{(\tau)}(\beta); \\ W_i(u) = u^{(i-n-1)}(\beta) - k_i u^{(\tau)}(\beta), \qquad n+1 \leqq i \leqq 2n, \\ \qquad\qquad i \neq n + \tau + 1; \qquad W_{n+\tau+1}(u) = u^{(\tau)}(\beta). \end{array}\right\} \quad (18)$$

Here $k, k_1, k_2, \cdots, k_{2n}$ are constants. (Given any solution u of the equation $L(u) = 0$ and any points α and β in (ab), it is clear that σ and τ exist so that corresponding constants k may be found in such way that $W_i = 0$ for all values i of the set $1, 2, \cdots, 2n$ except $i = n + \tau + 1$. It is this fact which suggests the use of this particular set of forms.) It is clear that these forms are linearly independent and of determinant unity. They are the special case of forms (11) in which

$$\left.\begin{array}{l} c_{ii} = 1; \qquad c_{i,\,\sigma+1} = -k_i, \qquad i \neq \sigma + 1 \text{ and } i \leqq n; \\ c_{\sigma+1,\,n+\tau+1} = -k; \qquad c_{i,\,n+\tau+1} = -k_i, \qquad i \neq n+\tau+1 \text{ and } i > n; \\ \qquad\qquad c_{ij} = 0 \text{ in all other cases.} \end{array}\right\} \quad (19)$$

Let $V_1, V_2, \cdots, V_{2n}$ be the forms which are adjoint to the W's in (18). By means of (16) we see that these forms are the special case of (15) in which

$$\left.\begin{array}{l} \gamma_{js} = r_{js}, \quad j, s = 1, 2, \cdots, 2n, \quad j \neq \sigma + 1, \quad j \neq n + \tau + 1; \\ \gamma_{\sigma+1,\,s} = \displaystyle\sum_{i=1}^{n} k_i r_{is}, \qquad s = 1, 2, \cdots, 2n, \qquad k_{\sigma+1} = 1; \\ \gamma_{n+\tau+1,\,s} = \displaystyle\sum_{i=n+1}^{2n} k_i r_{is} + k \sum_{i=1}^{n} k_i r_{is}, \qquad s = 1, 2, \cdots, 2n, \qquad k_{n+\tau+1} = 1. \end{array}\right\} \quad (20)$$

These values reduce to simpler forms for certain subscripts s in view of the fact that many of the quantities r_{ij} have the value zero, as we saw above.

Let us consider the special case in which $\sigma = n - 1$, $\tau = 0$, $k_i = 0$ $(i \neq n \text{ or } n + 1)$, $k = 0$. Then the conditions $W_\sigma = 0$, $\sigma = 1, 2, \cdots, n - 1, n + 1$, are

$$u(\alpha) = 0,\ u'(\alpha) = 0,\ \cdots,\ u^{(n-2)}(\alpha) = 0,\ u(\beta) = 0. \tag{21}$$

For this special case it is easy to see through either (20) or (12) that the adjoint conditions are equivalent to

$$y(\alpha) = 0,\ y(\beta) = 0,\ y'(\beta) = 0,\ \cdots,\ y^{(n-2)}(\beta) = 0, \tag{22}$$

the first of these being precisely $V_n = 0$ and the others on account of the relation $V_n = 0$ being equivalent to the remaining set of conditions $V_{n+2} = 0, \cdots, V_{2n} = 0$. In (21) and (22) we have a simple example of skew-self-adjoint boundary conditions. These conditions, with the exception of $y(\alpha) = 0$, are employed in the first theorem of § 4 and similar sets in the remaining theorems of that section.

When the reader has reached the first and second theorems of § 4 he will see that several analogous theorems may be obtained through the use of certain skew-self-adjoint conditions based on special cases of the forms (18) similar to those introduced in the last preceding paragraph.

§ 3. General theorems of comparison.

We are now in position readily to prove the following theorem:

FIRST GENERAL THEOREM OF COMPARISON. *Let u_1 be any solution of the equation $L(u) = 0$ which does not vanish in the interior of an interval $(\alpha\beta)$ lying in (ab); for definiteness say that it is positive* in the interior of $(\alpha\beta)$. Let y_1 be a solution of the equation $P(y) = 0$ so related to u_1 that*

$$R(u_1, y_1)\Big|_{x=\alpha}^{x=\beta} \leqq 0. \tag{23}$$

Suppose that the coefficients of $L(u)$ and $P(y)$ are so related that $p_k - m_k$ does not change sign in $(\alpha\beta)$ for any k of the set $0, 1, 2, \cdots, n$. Then, if p_k is different from m_k for some k of the set $0, 1, 2, \cdots, n$ and some x of $(\alpha\beta)$, at least one of the quantities $y_1^{(k)}$, $k = 0, 1, 2, \cdots, n$, for which the corresponding $p_k - m_k$ is different from zero at some point in $(\alpha\beta)$, vanishes in the interior of $(\alpha\beta)$, provided that, for every k for which $p_k - m_k \not\equiv 0$ in $(\alpha\beta)$, the two functions

$$p_k - m_k \text{ and } y_1^{(k)}$$

have the same sign at a point P_k in $(\alpha\beta)$.

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* It is obvious that we have a like result obtained by giving to u and y signs opposite to those used in the theorem.

The point P_k may well be taken at α or at β if $p_k - m_k$, when not identically zero in $(\alpha\beta)$, is different from zero at either of these points.

It is obvious that condition (23) may be replaced by

$$\sum_{i=1}^{2m} W_i(u_1)V_i(y_1) \leqq 0, \tag{24}$$

where $W_1, \cdots, W_{2n}$ and $V_1, \cdots, V_{2n}$ are any two sets of adjoint forms, in the sense of the definition in § 2. This condition is satisfied in particular if W_i and V_i have opposite signs when neither of them is zero. As a very special case, it is satisfied if u_1 and y_1 satisfy adjoint boundary conditions defined for the interval $(\alpha\beta)$ and with reference to the differential expression $L(u)$.

The proof of the foregoing theorem is based on equation (10), which for the case before us may be written in the form

$$\begin{aligned} R(u_1, y_1)\Big|_{x=\alpha}^{x=\beta} = \int_\alpha^\beta (p_n - m_n)y_1^{(n)}u_1dx \\ + \int_\alpha^\beta (p_{n-1} - m_{n-1})y_1^{(n-1)}u_1dx + \cdots \\ + \int_\alpha^\beta (p_1 - m_1)y_1'u_1dx + \int_\alpha^\beta (p_0 - m_0)y_1u_1dx. \end{aligned} \tag{25}$$

We now obtain the proof by showing that (25) yields a contradiction if we assume that no one of the quantities $y_1^{(k)}$, $k = 0, 1, 2, \cdots, n$, for which the corresponding $p_k - m_k$ is different from zero at some point in $(\alpha\beta)$, vanishes in the interior of $(\alpha\beta)$. Now by the hypothesis of the theorem each integrand in (25) whose factor $p_k - m_k$ is not identically zero in $(\alpha\beta)$ is positive at some point in $(\alpha\beta)$. From the hypothesis and the assumption just made it follows that no one of these integrands can change sign in $(\alpha\beta)$. Hence at least one of the integrals in (25) has a positive value and no one of them has a negative value. But this contradicts (23). On account of this contradiction we conclude to the truth of the theorem.

In a similar manner we prove the following:

SECOND GENERAL THEOREM OF COMPARISON. *Let y_2 be any solution of the equation $P(y) = 0$ for which $y_2^{(k)}$ and $p_k - m_k$ $(k = 0, 1, 2, \cdots, n)$ have the same sign in an interval $(\alpha\beta)$ of (ab) at every point for which neither is zero. Let u_2 be a solution of the equation $L(u) = 0$ which is positive at some point in $(\alpha\beta)$ and for which we have*

$$R(u_2, y_2)\Big|_{x=\alpha}^{x=\beta} \leqq 0 \qquad \left(\text{or } \sum_{i=1}^{2n} W_i(u_2)V_i(y_2) \leqq 0\right). \tag{26}$$

Then if p_k is not identically equal to m_k for every k of the set $0, 1, 2, \cdots, n$, u_2 vanishes in the interior of $(\alpha\beta)$.

This result follows at once from the relation obtained from (25) by replacing y_1 and u_1 throughout by y_2 and u_2, the method being to assume that u_2 does not vanish in the interior of $(\alpha\beta)$ and thus arrive at a contradiction of (26).

We point out some special cases of the foregoing theorems.

Note that the points α and β in the first of the theorems may be in particular two consecutive zeros of u_1.

In case $p_k \equiv m_k$ in $(\alpha\beta)$ for every k of the set $0, 1, 2, \cdots, n$, except $k = s$, it follows from the first of the foregoing general theorems that $y_1^{(s)}$ has a root in the interior of $(\alpha\beta)$. In particular, if $p_k \equiv m_k$ for $k = 1, 2, \cdots, n$, but $p_0 \not\equiv m_0$ in $(\alpha\beta)$ then y_1 has a root between α and β.

The last particular case of our general theorem contains as a very special instance the following classical theorem of Sturm:

Let u_1 be a solution of the equation $u'' + l_0 u = 0$. Denote by α and β two consecutive zeros of u_1. Let y_1 be a solution of the equation $y'' + p_0 y = 0$. Then y_1 vanishes between α and β provided that $p_0 \geqq l_0$ in $(\alpha\beta)$, the equality sign not holding throughout $(\alpha\beta)$.

If y_1 does not vanish between α and β it is of one sign between α and β. We shall therefore suppose that u_1 and y_1 are both positive between α and β and show that (23) then holds, a result which requires that y_1 vanish between α and β in accordance with the foregoing general theorem of comparison. (Observe that m_0 is now equal to l_0.) We have

$$R(u_1, y_1) = u_1' y_1 - u_1 y_1'.$$

Hence (23) reduces to $u_1'(\beta)y_1(\beta) - u_1'(\alpha)y_1(\alpha) \leqq 0$. That this inequality is now verified is easily shown by observing that $u_1'(\alpha)$ is positive and $u_1'(\beta)$ is negative while $y_1(\alpha)$ and $y_1(\beta)$ are positive or zero.

In the second of the foregoing theorems we may take α and β as two consecutive zeros of y_2 provided that $p_k \equiv m_k$ in $(\alpha\beta)$ for $k = 1, 2, \cdots, n$ but $p_0 \not\equiv m_0$ in $(\alpha\beta)$. The theorem asserts then that u_2 vanishes between the two consecutive zeros α and β of y_2. It is easy to formulate in a similar way less special cases of the same theorem.

§ 4. **Certain specific theorems of comparison.**

By means of the first general theorem of comparison we may readily prove the following:

FIRST SPECIFIC THEOREM OF COMPARISON. *Let α be a point in (ab) and let u_3 be a solution of the equation $L(u) = 0$ for which $u_3(\alpha) = 0$, $u_3'(\alpha) = 0$, $\cdots$, $u_3^{(n-2)}(\alpha) = 0$. Let β be the leftmost point (assumed existent) to the right of α in (ab) for which $u_3(\beta) = 0$. Let y_3 be a solution of the equation $P(y) = 0$ for which $y_3(\beta) = 0$, $y_3'(\beta) = 0$, $\cdots$, $y_3^{(n-2)}(\beta) = 0$.*

Then, if, for $k = 0, 1, 2, \cdots, n$, *we have*

$$p_k \geqq m_k \text{ or } p_k \leqq m_k \text{ in } (\alpha\beta),$$

according as k *is even or odd, the equality sign not holding throughout* $(\alpha\beta)$ *for every* k, *at least one of the functions* $y_3^{(k)}$, $k = 0, 1, 2, \cdots, n$, *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *has a root in the interior of* $(\alpha\beta)$, *except when* $p_k \equiv m_k$ *for* $k = 0, 1, 2, \cdots, n-1$, *in which case either* $y_3^{(n)}$ *vanishes between* α *and* β *or all the functions* y_3, y_3', y_3'', $\cdots$, $y_3^{(n-1)}$ *vanish between* α *and* β.

Under the hypothesis it is clear that $u_3^{(n-1)}(\alpha)$ and $y_3^{(n-1)}(\beta)$ are different from zero. We assume (as we may without loss of generality) that u_3 is positive in the neighborhood of α to the right and that y_3 is positive in the neighborhood of β to the left. Then it is clear that u_3' is positive near α to the right, and thence in turn that u_3'', u_3''', $\cdots$, $u_3^{(n-1)}$ are positive near α to the right. In a similar way it may be seen that y_3' is negative near β to the left, y_3'' is positive near β to the left, y_3''' is negative, $\cdots$, $y_3^{(n-1)}$ is positive or negative near β to the left according as n is odd or even.

In view of (4) and the facts just adduced we see from the first general theorem of comparison that the above specific theorem is true if

$$l_n(\alpha)y_3(\alpha)u_3^{(n-1)}(\alpha) \geqq 0,$$

and hence if $y_3(\alpha) \geqq 0$. But if $y_3(\alpha) < 0$, y_3 has a root in the interior of $(\alpha\beta)$; whence it follows readily that the functions y_3', y_3'', $\cdots$, $y_3^{(n-1)}$ all have roots in the interior of $(\alpha\beta)$. Hence the theorem is true in any case.

It is easy to see that we have the following corollary:

COROLLARY. *If* $p_k - m_k \equiv 0$ *in* $(\alpha\beta)$ *for* $k = s+1, s+2, \cdots, n$, $s < n$, *then* $y_3^{(s)}$ *has a root in the interior of* $(\alpha\beta)$.

Again from the first general theorem of comparison we may derive the following:

SECOND SPECIFIC THEOREM OF COMPARISON. *Let* u_4 *be a solution of the equation* $L(u) = 0$ *for which* $u_4(\beta) = 0$, $u_4'(\beta) = 0$, $\cdots$, $u_4^{(n-2)}(\beta) = 0$. *Let* α *be the rightmost point (assumed existent) to the left of* β *in* (ab) *for which* $u_4(\alpha) = 0$. *Let* y_4 *be a solution of the equation* $P(y) = 0$ *for which* $y_4(\alpha) = 0$, $y_4'(\alpha) = 0$, $\cdots$, $y_4^{(n-2)}(\alpha) = 0$. *Then, if, for* $k = 0, 1, 2, \cdots, n$, *we have*

$$p_k \geqq m_k \text{ or } p_k \leqq m_k \text{ in } (\alpha\beta)$$

according as n *is even or odd, the equality sign not holding throughout* $(\alpha\beta)$ *for every* k, *at least one of the functions* $y_4^{(k)}$, $k = 0, 1, \cdots, n$, *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *has a root* *in the interior of* $(\alpha\beta)$, *except when* $p_k \equiv m_k$ *for* $k = 0, 1, \cdots, n-1$, *in which case either* $y_4^{(n)}$ *vanishes between* α *and* β *or all the functions* $y_4, y_4', \cdots, y_4^{(n-1)}$ *vanish between* α *and* β.

Under the hypothesis it is clear that $u_4^{(n-1)}(\beta)$ and $y_4^{(n-1)}(\alpha)$ are different from zero. We assume (as we may without loss of generality) that u_4 is positive near β to the left and that y_4 is positive or negative near α to the right according as n is even or odd. Then $y_4', y_4'', \cdots, y_4^{(n-1)}$ are positive or negative near α to the right according as n is even or odd. Also, $u_4^{(k)}$, $k = 0, 1, \cdots, n-1$, is positive or negative near β to the left according as k is even or odd.

In view of these facts and equation (4) we see from the first general theorem of comparison that the foregoing specific theorem is true if

$$l_n(\beta) y_4(\beta) u_4^{(n-1)}(\beta) \leqq 0,$$

and hence if $y_4(\beta) \leqq 0$ when n is odd and if $y_4(\beta) \geqq 0$ when n is even. But if $y_4(\beta) > 0$ and n is odd or if $y_4(\beta) < 0$ and n is even, it is clear that y_4 has a zero in the interior of $(\alpha\beta)$; whence it follows readily that the functions $y_4', y_4'', \cdots, y_4^{(n-1)}$ vanish in the interior of $(\alpha\beta)$. Hence the theorem is true in any case.

COROLLARY. *If* $p_k - m_k \equiv 0$ *in* $(\alpha\beta)$ *for* $k = s+1, s+2, \cdots, n$, $s < n$, *then* $y_4^{(s)}$ *has a root in the interior of* $(\alpha\beta)$.

It is clear that other particular theorems analogous to the foregoing may readily be obtained in a similar manner through the use of skew-self-adjoint boundary conditions in which $n-1$ of the n conditions involve a single point. (Compare the last remark in § 3.)

From the second general theorem of comparison one readily derives the following result:

THIRD SPECIFIC THEOREM OF COMPARISON. *Let* α *be a point in* (ab) *and* y_5 *be a solution of the equation* $P(y) = 0$ *for which* $y_5(\alpha) = 0, y_5'(\alpha) = 0, \cdots, y_5^{(n-2)}(\alpha) = 0$. *Let* β *be the leftmost point (assumed existent) to the right of* α *in* (ab) *for which* Y *vanishes, where* Y *is the product of those functions* $y_5^{(k)}$ $(k = 0, 1, 2, \cdots, n)$ *for which the corresponding difference* $p_k - m_k$ *is not identically zero in* $(\alpha\beta)$, *the assumption being that such a product is existent. Furthermore, let every such difference* $p_k - m_k$ *(when not zero) be throughout* $(\alpha\beta)$ *of the same sign* as* $y_5^{(n-1)}(\alpha)$. *Let* u_5 *be a solution of the equation* $L(u) = 0$ *which is positive at some point in* $(\alpha\beta)$ *and satisfies the condition*

$$R(u_5, y_5)\Big|_{x=\alpha}^{x=\beta} \leqq 0 \qquad \left[\text{or } \sum_{i=1}^{2n} W_i(u_5) V_i(y_5) \leqq 0\right].$$

Then this function u_5 *vanishes in the interior of* $(\alpha\beta)$.

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* It should be observed that this sign is determined by the other conditions on y to within a simultaneous change of sign of all the quantities $y^{(k)}$, $k = 0, 1, 2, \cdots, n$.

For the proof it is sufficient to observe that the functions $y_5^{(k)}$ entering into the product Y have in the interior of $(\alpha\beta)$ the same sign as $y_5^{(n-1)}(\alpha)$.

In a similar manner the following result is obtained:

FOURTH SPECIFIC THEOREM OF COMPARISON. *Let β be a point in (ab) and let y_6 be a solution of the equation $P(y) = 0$ for which $y_6(\beta) = 0$, $y_6'(\beta) = 0, \cdots, y_6^{(n-2)}(\beta) = 0$. Let α be the rightmost point (assumed existent) to the left of β in (ab) for which Y_1 vanishes, where Y_1 is the product of those functions $y_6^{(k)}$ $(k = 0, 1, 2, \cdots, n)$ for which the corresponding difference $p_k - m_k$ is not identically zero in $(\alpha\beta)$, the assumption being that such a product is existent. Furthermore let every such difference $p_k - m_k$ (when not zero) be throughout $(\alpha\beta)$ of the same sign* as y_6^k in the neighborhood of β to the left. Let u_6 be a solution of the equation $L(u) = 0$ which is positive at some point in $(\alpha\beta)$ and satisfies the condition*

$$R(u_6, y_6)\Big|_{x=\alpha}^{x=\beta} \leqq 0 \qquad \left[\text{or } \sum_{i=1}^{2n} W_i(u_6)V_i(y_6) \leqq 0\right].$$

Then this function u_6 vanishes in the interior of $(\alpha\beta)$.

§ 5. Concerning a case in which P(y) has special form.

By means of the second general theorem of comparison we may readily prove the following:

THEOREM. *Let α and β be two consecutive zeros in (ab) of a solution y_7 of the equation $P(y) = 0$ and suppose that each of these zeros is of order $n - 1$. Suppose further that the products $y_7^{(k)}$ $(p_k - m_k)$, $k = 0, 1, 2, \cdots, n$, are all non-negative or all non-positive in $(\alpha\beta)$. Then any solution u_7 of the equation $L(u) = 0$ vanishes between α and β provided that $p_k - m_k$ is not identically zero in $(\alpha\beta)$ for every k of the set $0, 1, 2, \cdots, n$ and provided that $p_0 - m_0$ is negative for some point in $(\alpha\beta)$ when n is even or*

$$l_n(\beta)y_7^{(n-1)}(\beta)u_7(\beta) \leqq l_n(\alpha)y_7^{(n-1)}(\alpha)u_7(\alpha), \tag{27}$$

when n is odd.

We shall prove this by supposing that u_7 does not vanish between α and β and thus obtaining a contradiction. We then take u_7 to be positive between α and β. We choose the sign of y_7 so that the products $y_7^{(k)}(p_k - m_k)$ are all non-negative in $(\alpha\beta)$. We have

$$\begin{aligned} R(u_7, y_7)\Big|_{x=\alpha}^{x=\beta} &= (-1)^{n-1} l_n y_7^{(n-1)} u_7 \Big|_{x=\alpha}^{x=\beta} \\ &= (-1)^{n-1}[l_n(\beta)y_7^{(n-1)}(\beta)u_7(\beta) - l_n(\alpha)y_7^{(n-1)}(\alpha)u_7(\alpha)]. \end{aligned} \tag{28}$$

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* It should be observed that this sign is determined by the other conditions on y to within a simultaneous change of sign of all the quantities $y^{(k)}$, $k = 0, 1, 2, \cdots, n$.

From the second general theorem of comparison it follows that for the conclusion of the theorem it is sufficient that the last member of (28) shall be not greater than zero. When n is odd this condition is necessitated by (27). That the condition is satisfied when n is even follows from the fact that we then have $y_7^{(n-1)}(\alpha)$ negative and $y_7^{(n-1)}(\beta)$ positive, the latter statements being true as one sees through the fact that y_7 is negative between α and β and has at these points zeros of order $n - 1$.

It is clear that the foregoing theorem is best suited to applications to equations of even order. We shall indicate the character of its usefulness by applying it to a special form of the self-adjoint equation of the fourth order, namely,

$$u'''' + \lambda u'' + \lambda^2 \varphi u = 0, \tag{29}$$

where φ is a function of x which is continuous in (ab) and λ is a real parameter independent of x.

As an equation of comparison we take

$$y'''' + 10k^2 y'' + 9k^4 y = 0, \qquad k = \text{const.}, \tag{30}$$

which has the solution $y = \cos^3 k(x - c)$. In order that the theorem shall apply we must have

$$10k^2 = \lambda, \qquad 9k^4 - \lambda^2 \varphi \leqq 0. \tag{31}$$

Therefore we assume that

$$\varphi \geqq \tfrac{9}{100} \text{ in } (ab),$$

so that both conditions (31) are satisfied when $10k^2 = \lambda$. For $x > b$ we take for $\varphi(x)$ the value $\varphi(b)$.

Applying the theorem of this section, we now see that any solution $u(x, \lambda)$ of (29) must have a zero in every interval of length π/k or $\pi\sqrt{10}/\sqrt{\lambda}$ throughout which $x \geqq a$. Hence as λ increases indefinitely the number of roots of $u(x, \lambda)$ in (ab) increases indefinitely.

This example is sufficient to illustrate the nature of the theorem of this section as regards its use in treating oscillation problems for certain interesting special cases of differential equations of any order. It suggests an investigation the carrying out of which the author reserves for a future occasion.

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